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CONTROLLABILITY AND OBSERVABILITY IN THE PROBLEM OF STABILIZING STEADY MOTIONS OF NON-HOLONOMIC MECHANICAL SYSTEMS WITH CYCLIC COORDINATES†

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The approach to the solution of stabilization problems for steady motions of holonomic mechanical systems [1, 2] based on linear control theory, combined with the theory of critical cases of stability theory, is used to solve the analogous problems for non-holonomic systems. It is assumed that the control forces may affect both cyclic and positional coordinates, where the number r of independent control inputs may be considerably less than the number n of degrees of freedom of the system, unlike in many other studies (see, e.g., [3–5]), in which as a rule $r = n$. Several effective new criteria of controllability and observability are formulated, based on reducing the problem to a problem of less dimension. Stability analysis is carried out for the trivial solution of the complete non-linear system, closed by a selected control. This analysis is a necessary step in solving the stabilization problem for steady motion of a non-holonomic system (unlike holonomic systems), since in most cases such a system is not completely controllable.

Some stabilization problems for steady motions of non-holonomic systems have been considered previously [6–8], but questions of controllability and observability were not investigated.

1. FORMULATION OF THE STABILIZATION PROBLEM FOR STEADY MOTIONS OF NON-HOLONOMIC MECHANICAL SYSTEMS

Consider a non-holonomic mechanical system whose position is defined by generalized coordinates $q_1, \ldots, q_n$. The velocities $\dot{q}_1, \ldots, \dot{q}_n$ are subject to $n - l(l < n)$ time-independent non-holonomic constraints

$$\dot{q}_\chi = \sum_{r=1}^{l} b_{\chi r}(q)\dot{q}_r \tag{1.1}$$

Here, are throughout this paper, unless otherwise indicated, the subscripts take the following values: $i = 1, \ldots, k$; $p, r, s = 1, \ldots, l$, $\alpha, \beta, \gamma = k + 1, \ldots, l$; $\mu = m + 1, \ldots, n$; $\rho = l + 1, \ldots, m$; $\chi = l + 1, \ldots, n$.

The equations of motion of a non-holonomic mechanical system will be taken in the form of the Voronets equations [9]

$$\frac{d}{dt}\frac{\partial\theta}{\partial\dot{q}_r} - \frac{\partial\theta}{\partial q_r} - \sum_{\chi=l+1}^{n}\frac{\partial\theta}{\partial q_\chi}b_{\chi r} - \sum_{\chi=l+1}^{n}\theta_\chi\sum_{s=1}^{l}\nu_{\chi rs}\dot{q}_s = Q_r + \sum_{\chi=l+1}^{n}Q_\chi b_{\chi r} \tag{1.2}$$

Where θ and θ_χ are the results of eliminating the quantities $\dot{q}_\chi$, using Eqs (1.1), from the expressions for T and $\partial T/\partial\dot{q}_\chi$, where T is the kinetic energy of the system

$$T = \frac{1}{2}\sum_{r,s=1}^{n} a_{rs}(q)\dot{q}_r\dot{q}_s > 0$$

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$$\nu_{\chi rs}=\frac{\partial b_{\chi r}}{\partial q_s}-\frac{\partial b_{\chi s}}{\partial q_r}-\sum_{\chi'=l+1}^{n}\left(b_{\chi' r}\frac{\partial b_{\chi s}}{\partial q_{\chi'}}-b_{\chi' s}\frac{\partial b_{\chi r}}{\partial q_{\chi'}}\right)$$

$$\theta_{\chi p}=a_{\chi p}+\sum_{\chi'=l+1}^{n}a_{\chi\chi'}b_{\chi' p}$$

and Q_r and Q_χ are generalized forces corresponding to the generalized coordinates q_r and q_χ.

Equations (1.2), together with Eqs (1.1), constitute a closed system of order $n + l$ in the variables $q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}$.

Let us assume that the following conditions are satisfied [10]

$$\frac{\partial T}{\partial q_\mu}=0,\quad Q_\mu=0,\quad \frac{\partial b_{\chi r}}{\partial q_\mu}=0,\quad Q_{r\rho}=Q_{r\rho}(q_r,\dot{q}_r,q_\rho)$$

implying that the last $n-m$ equations of non-holonomic constraints (1.1) are Chaplygin-type constraints (the first $m-l$ constraints are of general type).

In addition, suppose some of the generalized coordinates $q_1, \ldots, q_n$ of the mechanical system are cyclic coordinates (CCs). It should be mentioned that, while the definition of CCs for holonomic conservative systems automatically guarantees the existence of cyclic integrals in steady motions (SMs), there are several definitions of CCs for non-holonomic systems [10]. In that situation, moreover, the equations of motion of the system may not have cyclic integrals, though they may admit of SMs. In this paper we will adopt the definition of CCs in [10], which guarantees the existence of SMs. We shall assume that the coordinates q_α are cyclic in the sense of that definition, that is,

$$\frac{\partial\theta}{\partial q_\alpha}=0,\quad \frac{\partial b_{\rho r}}{\partial q_\alpha}=0,\quad \frac{\partial}{\partial q_\alpha}\sum_{\chi=l+1}^{n}\theta_{\chi p}\nu_{\chi rs}=0,\quad Q_{i,\alpha,\rho}=Q_{i,\alpha,\rho}(q_i,\dot{q}_i,\dot{q}_\alpha,q_\rho)$$

The remaining coordinates q_i and q_ρ are positional.

Let us assume that the generalized forces corresponding to the positional coordinates are sums of potential, dissipative and control forces; the only generalized forces affecting part of the CCs ($\alpha = k+1, \ldots, k+h$) are the control forces; no generalized forces at all affect the other CCs ($\beta = k+h+1, \ldots, l$). Note that if the control forces affect all the CCs, then $h = l-k$. Thus, the generalized forces may be expressed in the form

$$Q_i=\frac{\partial U}{\partial q_i}-\frac{\partial\Phi}{\partial\dot{q}_i}+F_i,\quad Q_\rho=\frac{\partial U}{\partial q_\rho}+F_\rho,\quad Q_\alpha=F_\alpha,\quad Q_\beta=0$$

where U is the force function, Φ is the reduced dissipative function, and the control forces F_j depend on the generalized coordinates q_i and q_ρ and depend linearly on the control inputs $u^{(1)}(r_1\times 1)$, $u^{(2)}(r_2\times 1)$, $u^{(3)}(r_3\times 1)$, applied along the coordinates q_i, q_ρ and q_α, respectively. Depending on the particular problem under consideration, the controls $u^{(1)}$, $u^{(2)}$, $u^{(3)}$ may be introduced in various ways.

Information about the values of q_i, $\dot{q}_i$, $\dot{q}_\alpha$ and q_ρ is obtained by gauges mounted both on the system itself and outside it. The $(s\times 1)$-dimensional vector of measurements is generally a function of all the positional coordinates q_i and q_ρ and of the cyclic and independent positional velocities $\dot{q}_\alpha$ and $\dot{q}_i$.

Suppose that under certain initial conditions the system may have a steady motion such that the positional coordinates and cyclic velocities are constant:

$$q_i(t)=q_{i0},\quad \dot{q}_i(t)=0,\quad \dot{q}_\alpha(t)=\dot{q}_{\alpha 0}=\omega_\alpha,\quad q_\rho(t)=q_{\rho 0} \tag{1.3}$$

In such a case the m constant quantities q_{i0}, $\dot{q}_{\alpha 0}$ and $\dot{q}_{\rho 0}$ will satisfy m equations, which will not be written out here.

Consequently, in the general case the SM may turn out to be isolated. In some cases, given various additional assumptions on the coefficients of the equations of the non-holonomic constraints, the kinetic energy and the generalized forces, a manifold of SMs may exist [9, 10]. We shall assume that the control forces vanish on a SM.

The stabilization problem for a SM is as follows: By a suitable choice of the control inputs, applied along both cyclic and positional coordinates (or part of those coordinates), it is required to make SM (1.3) asymptotically stable (or simply stable) with respect to the positional coordinates q_i and q_ρ, positional velocities $\dot{q}_i$ and cyclic velocities $\dot{q}_\alpha$. The optimal stabilization problem for a SM may be formulated in analogous terms. The special feature of the formulation of the stabilization problem for SMs of non-holonomic systems (as of the stability problem) is that the constraint equations occur in the structure of the system of equations of perturbed motion (except in the case of Chaplygin systems).

We now introduce the errors

$$x_i = q_i - q_{i0}, \quad y_\alpha = \dot{q}_\alpha - \omega_\alpha, \quad z_\rho = q_\rho - q_{\rho 0} \tag{1.4}$$

and with the equations of perturbed motion on the basis of Eqs (1.1) and (1.2) in terms of the variables $x(k \times 1), y((l-k) \times 1), z((m-l) \times 1)$, with the linear terms written separately

$$\begin{aligned}
&A\ddot{x} + C\dot{y} = W_1 x + D_1 \dot{x} + P_1 y + V_1 z + F^{(1)}u^{(1)} + D_3^T F^{(2)} u^{(2)} + X(x, \dot{x}, y, z)\\
&C^T\ddot{x} + B\dot{y} = W_2 x + D_2 \dot{x} + P_2 y + V_2 z + F^{(3)}u^{(3)} + P_3^T F^{(2)} u^{(2)} + Y(x, \dot{x}, y, z)\\
&\dot{z} = W_3 x + D_3 \dot{x} + P_3 y + V_3 z + Z(x, \dot{x}, y, z)
\end{aligned} \tag{1.5}$$

The formulae for the elements of the matrices $A, B, \ldots$ are similar to the appropriate formulae in previous studies [10]; X, Y and Z are vector-functions containing terms of order greater than one in the newly introduced variables.

System (1.5) has the most general structure, special cases of which are the equations of perturbed motions in a neighbourhood of!

1. the equilibrium positions of a holonomic system ($k = l = n = m, y \equiv 0, z \equiv 0$);
2. the equilibrium positions of a non-holonomic system with constraints of general form ($l = k$, $m = n, y \equiv 0$);
3. SMs of holonomic systems ($l = n, z \equiv 0$);
4. SMs of non-holonomic Chaplygin systems ($m = l, z \equiv 0$).

Under specific conditions, which arise in different problems, the structure of the equations of perturbed motion (1.5) may be simplified considerably. The additional conditions most commonly adopted in analysing the stability of SMs of non-holonomic systems, when there are no control forces, are the following [10].

$$b_{\rho\alpha} = 0 \text{ (Condition 1)}, \qquad \sum_{\mu=m+1}^{n} \theta_{\mu\beta} v_{\mu\alpha\gamma} = 0 \text{ (Condition 2)} \tag{1.6}$$

If condition 1 is satisfied, then P_2, P_3, W_3 and V_3 in Eqs (1.5) are zero matrices, while the zero matrices in the case when Condition 2 is satisfied are P_2 and P_3. But if both Conditions 1 and 2 hold, the equations corresponding to CCs and the equations corresponding to the equations of the non-holonomic constraints do not contain linear terms in the variables x_i, y_α and z_ρ, that is, $P_j = 0$, $W_j = 0$, $V_j = 0$ ($j = 2, 3$).

The equation of measurements may be written in the form

$$\sigma = H_1 x + H_2 \dot{x} + H_3 y + H_4 z \tag{1.7}$$

where $\sigma(s \times 1)$ is the linear part of the measurement vector and $H_1, \ldots, H_4$ are constant matrices of the appropriate dimensions.

The linearized equations of motion in the form (1.5) and the equation of measurements (1.7) are fundamental for solving the stabilization problem for steady motions of non-holonomic systems with time-independent constraints.

The solution of the stabilization problem [1, 2] involves:

1. determining the basic possibilities of stabilization, which reduces to investigating the controllability of system (1.5);
2. determining the rational composition of measurement information on the state of the system (the quantities $x, y, \dot{x}$ and z), necessary in order to construct a stabilizing control, which reduces to analysing the observability of system (1.5), (1.7);
3. constructing the stabilization algorithm itself, e.g., by estimating the state vector of the system, which is constructed using the previously determined measurement information [1];
4. analysing the stability of the non-linear system closed by a selected linear control.

2. CONTROLLABILITY

For direct application of known controllability criteria [13, 14] to system (1.5), written in Cauchy form, one must investigate the ranks of complicated matrices of high orders.

To investigate the controllability of system (1.5), we introduce variables

$$\eta = By + C^T\dot{x} - (D_2 - P_2B^{-1}C^T)x, \quad \zeta = z - (D_3 - P_3B^{-1}C^T)x$$

and transform the linearized system (1.5) to the form

$$S\ddot{x} + N\dot{x} + Kx + M_1\eta + M_2\zeta = F^{(1)}u^{(1)} + D_4^TF^{(2)}u^{(2)} - CB^{-1}F^{(3)}u^{(3)}$$
$$\dot{\eta} = R_1x + P_2B^{-1}\eta + V_2\zeta + P_3^TF^{(2)}u^{(2)} + F^{(3)}u^{(3)} \qquad (2.1)$$

$$\dot{\zeta} = R_2x + P_3B^{-1}\eta + V_3\xi \qquad (2.2)$$

where

$$S = S^T > 0, \quad S = A - CB^{-1}C^T, \quad D_4 = D_3 - P_3B^{-1}C^T, \quad D_5 = D_2 - P_2B^{-1}C^T$$
$$N = CB^{-1}D_5 + P_1B^{-1}C^T - D_1, \quad K = -W_1 - P_1B^{-1}D_5 + CB^{-1}R_1 - V_1D_4$$
$$M_1 = (CB^{-1}P_2 - P_1)B^{-1}, \quad M_2 = CB^{-1}V_2 - V_1$$
$$R_1 = W_2 + V_2D_3 + P_2B^{-1}D_5, \quad R_2 = W_3 + V_3D_4 + P_3B^{-1}D_5$$

Using the controllability criteria of [14] one can prove the following.

Theorem 1. System (2.1), (2.2) is controllable if and only if

$$\operatorname{rank}\begin{Vmatrix} L_1(\lambda) & M_1 & M_2 & F^{(1)} & D_4^TF^{(2)} - & CB^{-1}F^{(3)} \\ -R_1 & \lambda E_{l-k} - P_2B^{-1} & -V_2 & 0 & P_3^TF^{(2)} & F^{(3)} \\ -R_2 & -P_3B^{-1} & \lambda E_{m-l} - V_3 & 0 & 0 & 0 \end{Vmatrix} = m, \quad \forall\lambda \in \Lambda$$

$$\Lambda = \{\lambda_i : \det[L(\lambda)] = 0\}, \quad L_1(\lambda) = S\lambda^2 + N\lambda + K$$

$$L(\lambda) = \begin{Vmatrix} L_1(\lambda) & M_1 & M_2 \\ -R_1 & \lambda E_{l-k} - P_2B^{-1} & -V_2 \\ -R_2 & -P_3B^{-1} & \lambda E_{m-l} - V_3 \end{Vmatrix}$$

Note that controllability of system (2.1), (2.2) (that is, satisfaction of the conditions of Theorem 1) may be achieved, generally speaking, when only one of the control inputs $u^{(1)}$, $u^{(2)}$, $u^{(3)}$ is present.

If conditions (1.6) are satisfied, the special features of the structure of system (1.5) enables one to obtain new, effective controllability criteria by reducing the problem to systems of lower order.

Controllability of system (1.5) when Condition 1 is satisfied. It follows from the structure of Eqs (2.1) and (2.2) that, if Condition 1 of (1.6) is satisfied, system (1.5) splits into two subsystems ($R_2 = 0$), the second of which, corresponding to the equations of the non-holonomic constraints, is uncontrollable. Thus, given any control forces, the system is uncontrollable with respect to the variables ζ, which correspond to the positional variables whose velocities are dependent because of the constraint equations. This is a special feature of systems with non-holonomic constraints compared with holonomic constraints.

Application of the controllability criterion of [14] to subsystem (2.1) (with $\zeta \equiv 0$), allowing for the special structure of the system, and the use of equivalent transformations enable one, via reduction, to prove the following theorem, in which the conditions for controllability of a system of order $k + l$ reduce to verification of ranks of matrices of order k.

Theorem 2. System (2.1) of order $k + l$ is controllable if and only if the following conditions are satisfied

$$\operatorname{rank}\begin{Vmatrix} K & P_1B^{-1} & F^{(1)} & D_3^TF^{(2)} & CB^{-1}F^{(3)} \\ -R_1 & 0 & 0 & 0 & F^{(3)} \end{Vmatrix} = k, \quad \text{if} \quad \lambda = 0 \in \Lambda_2$$

$$\operatorname{rank}\| L_1(\lambda),\ F^{(1)},\ D_3^TF^{(2)},\ CB^{-1}F^{(3)} \| = k, \quad \forall\lambda \neq 0 \in \Lambda_2$$

$$\left(\Lambda_2 = \{\lambda_i : \det[L_2(\lambda)] = 0\}, \quad L_2(\lambda) = \det\begin{Vmatrix} L_1(\lambda) & -P_1B^{-1} \\ -R_1 & \lambda E_{l-k} \end{Vmatrix} \right)$$

Controllability of system (1.5) when both Conditions 1 and 2 are satisfied.
Suppose both conditions of (1.6) are satisfied. Let us split the vector η into parts

$$\eta = \left\| \eta_1^T \eta_2^T \eta_3^T \right\|^T \eta_1(h_1 \times 1),\ \eta_2((h-h_1)\times 1),\ \eta_3((l-k-h)\times 1)$$

depending on which CCs are subject to control inputs and which are not. Here h_1 is the number of control inputs applied along the CCs. We represent the matrix $F^{(3)}((l-k)\times h_1)$ in the form

$$\operatorname{col}\left\{ E_{h_1},\ F^{(4)}_{(h-h_1)\times h_1},\ 0 \right\}$$

where $E_{h1}(h_1 \times h_1)$ is the identity matrix.

Note that if the control inputs affect all the CCs and they are all independent, then $F^{(3)}$ is the identity matrix of order $l-k$.

Introducing a new variable $\xi = \eta_2 - F^{(4)}\eta_1$, we write Eqs (2.1) in the form

$$S\dddot{x} + N\ddot{x} + K\dot{x} - P_1B^{-1}F^{(3)}\eta_1 = F^{(1)}u^{(1)} + D_3^TF^{(2)}u^{(2)} + CB^{-1}F^{(3)}u^{(3)}, \quad \dot{\eta}_1 = u^{(3)} \tag{2.3}$$

$$\dot{\xi} = 0, \quad \dot{\eta}_3 = 0 \tag{2.4}$$

Obviously, subsystem (2.4) is uncontrollable, the corresponding roots of the characteristic equation being zero. The influence of the control inputs $u^{(3)}$ applied along the CCs η on controllability with respect to positional coordinates depends essentially on the nature of the matrices C and P_1.

The case $P_1 \equiv 0$, $C \not\equiv 0$. Using the previously proposed criterion of [14] and taking the structure of the system into account, together with the spectrum of its eigenvalues

$$\lambda \in \Lambda_1, \lambda = 0; \quad \Lambda_1 = \{\lambda_i : \det[L_1(\lambda)] = 0\}$$

one can prove the following theorems.

Theorem 3. System (2.3) of order $2k + h_1$ with $P_1 \equiv 0$ is controllable if and only if the following system of order $2k$ is controllable

$$S\dddot{x} + N\ddot{x} + K\dot{x} = F^{(1)}u^{(1)} + D_3^TF^{(2)}u^{(2)} - CB^{-1}F^{(3)}u^{(3)}$$

and moreover

$$\operatorname{rank}\left\| K, F^{(1)},\ D_3^TF^{(2)} \right\| = k$$

Theorem 4. System (2.3) of order $2k + h_1$ with $P_1 \equiv 0$ is controllable if and only if

$$\text{(a) } \operatorname{rank}\left\| K, F^{(1)}, D_3^TF^{(2)} \right\| = k, \quad \text{(b) } \operatorname{rank}\left\| L_1(\lambda), CB^{-1}F^{(3)}, F^{(1)}, D_3^TF^{(2)} \right\| = k, \quad \forall\lambda \neq 0 \in \Lambda_1$$

For one of the most common formulations of the control problem for a system with CCs [15], when the controls are applied only along CCs (or some of them), we have the following corollary.

Corollary 1. If the control inputs act only along CCs ($F^{(1)} \equiv 0$, $F^{(2)} \equiv 0$), then system (2.3) of order $2k + h_1$ with $P_1 \equiv 0$ is controllable if and only if

$$\det K \neq 0 \quad \text{and} \quad \operatorname{rank}\left\| L_1(\lambda),\ CB^{-1}F^{(3)} \right\| = k, \quad \forall\lambda \neq 0 \in \Lambda_1$$

The case $P_1 \equiv 0, C \equiv 0$. In this case system (2.3) splits into two independent subsystems, the second of which is completely controllable.

Corollary 2. In the case when $P_1 \equiv 0$, $C \equiv 0$, system (2.3) of order $2k + h_1$ is controllable if and only if

(a) $\operatorname{rank}\,[L_1(\lambda), F^{(1)}, D_3^T F^{(2)}] = k, \quad \forall\lambda \in \Lambda_1$ or

(b) the control inputs are applied along all positional coordinates whose velocities are independent because of the constraint equations ($F^{(1)} = E_k$).

Thus, a necessary condition for controllability of system (2.3) in the case when $P_1 \equiv 0$, $C \equiv 0$, is that the control inputs affect the positional coordinates, and moreover the controllability may be achieved by applying the controls both to those positional coordinates whose velocities are independent, and to those whose velocities are dependent because of the constraint equations.

The case $P_1 \neq 0, C \equiv 0$. In this case system (2.3) may be considered as a system consisting of subsystems connected in series, one of which is obviously controllable. Then, introducing an auxiliary control vector w of order h_1, we can prove the following theorem.

Theorem 5. A necessary and sufficient condition for system (2.3) of order $2k + h_1$ in the case when $C \equiv 0$ to be controllable is that the following system of order $2k$ be controllable

$$S\ddot{x} + N\dot{x} + Kx = F^{(1)}u^{(1)} + D_3^T F^{(2)}u^{(2)} + P_1 B^{-1}F^{(3)}w \tag{2.5}$$

Using previous results [16], one can show that system (2.5) is controllable if and only if

$$\operatorname{rank}\left\| L_1(\lambda), F^{(1)}, D_3^T F^{(2)}, P_1 B^{-1}F^{(3)} \right\| = k, \quad \forall\lambda \in \Lambda_1$$

The case $P_1 \neq 0, C \neq 0$. In this case it cannot be proved that the question of whether system (2.3) is controllable can be reduced to analysing a system of lower order, as in Theorem 3. The following theorem may be proved by arguments similar to those used to prove Theorem 4.

Theorem 6. System (2.3) of order $2k + h_1$ is controllable if and only if

$$\operatorname{rank}\left\| L_1(\lambda), F^{(1)}, D_3^T F^{(2)}, (\lambda C - P_1)B^{-1}F^{(3)} \right\| = k, \quad \forall\lambda \in \Lambda_1, \lambda = 0$$

Now, using Theorem 6, one can prove the following reduction theorem.

Theorem 7. System (2.3) of order $2k + h_1$ is controllable if and only if the system

$$S\ddot{x} + N^T\dot{x} + K^T x = 0$$

of order $2k$ is observable by measurements

$$\sigma = H_1 x + H_2\dot{x}; \quad H_1 = \left\| \begin{matrix} F^{(1)T} \\ F^{(2)T}D_3 \\ -F^{(3)T}B^{-1}P_1^T \end{matrix} \right\|, \quad H_2 = \left\| \begin{matrix} 0 \\ 0 \\ -F^{(3)T}B^{-1}C^T \end{matrix} \right\|$$

Corollary. If the control inputs act only along all the CCs and they are independent ($F^{(1)} = 0$, $F^{(2)} = 0$, $F^{(3)} = E_{l-k}$), then system (2.3) of order $2k + h_1$ is controllable if and only if

$$\operatorname{rank}\left\| L_1(\lambda), C\lambda - P_1 \right\| = k, \quad \forall\lambda \in \Lambda_1$$

Similar theorems concerning reduction of the controllability problem and controllability criteria may be formulated when Condition 2 is satisfied.

3. OBSERVABILITY

In order to solve the stabilization problem for non-holonomic mechanical systems by designing controls in the form of linear feedback, one must have information on the state of the system, obtained from various measuring instruments. The question of the minimum amount of accessible measurement information necessary to determine the full state vector of the system is of practical interest. It should be borne in mind that positional coordinates, positional velocities and cyclic velocities are measured by different instruments.

If Conditions 1 and 2 are satisfied, the special structure of the system makes it possible, as in controllability analysis, to reduce the problem, thereby obtaining new, simpler and effective observability criteria.

Here we will limit ourselves to observability conditions for system (1.5) in the case when Conditions 1 and 2 are satisfied by measurements

$$\sigma_1 = H_1 x + H_2 \dot{x}, \quad \sigma_1(s_1 \times 1) \tag{3.1}$$

$$\sigma_2 = H_3 y, \quad \sigma_2(s_2 \times 1) \tag{3.2}$$

It is assumed that the matrices H_1 $(s_1 \times k)$, $H_2(s_1 \times k)$, $H_3(s_2 \times (l-k))$ are of full rank.

Theorem 8. System (1.5), (3.1) is observable if and only if the following conditions are satisfied.

$$\operatorname{rank}\begin{Vmatrix} L_1(\lambda) \\ H_1 + \lambda H_2 \end{Vmatrix} = k, \quad \forall\lambda \in \Lambda_1; \ \operatorname{rank} H_1 = k, \ \operatorname{rank}\begin{Vmatrix} P_1 B^{-1} & V_1 \end{Vmatrix} = m-k \tag{3.3}$$

For the proof, it is convenient, as at the beginning of Section 2 to introduce variables ζ and η, as well as a variable $\chi = P_1 B^{-1}\eta + V_1\zeta$, and to express system (1.5) in the form

$$S\ddot{x} + N\dot{x} + Kx - \chi = 0, \quad \dot{\chi} = 0$$

It is important to stress that a necessary condition for conditions (3.3) to hold is that $s_1 = k$ (the number s_1 of measurements (3.3) equals the number of positional coordinates x); that is, all the positional coordinates must be measured. A necessary condition for observability of the vector χ to imply observability of the variables η and ζ, is that $k \geqslant m-k$, i.e. the number k of positional coordinates must be not less than the sum of the number of cyclic coordinates and the number of non-holonomic constraints of general type.

It is obvious that system (1.5) is non-observable by measurement of only positional velocities ($H_1 \equiv 0$). It can be shown that system (1.5) cannot be completely observable even by measurement (3.2).

Remark. If the stabilization problem for steady motions of a non-holonomic mechanical system is limited to the achievement of non-asymptotic stability, there is no need to estimate the entire state vector. It may therefore be convenient to estimate only that part of the vector of cyclic velocities (see Sections 2) affected by the control input.

If σ_1 and σ_2 are being measured, the first and second observability conditions of Theorem 8 are retained, but the third becomes

$$\operatorname{rank}\begin{Vmatrix} P_1 B^{-1} & V_1 \\ H_3 & 0 \end{Vmatrix} = m-k$$

4. AN ALGORITHM FOR THE STABILIZATION OF STEADY MOTIONS AND INVESTIGATION OF STABILITY OF THE CLOSED SYSTEM

In the general case, the initial linearized system (1.5), (1.7) may be controllable and observable. One can then construct a feedback control based on estimation of the state vector in such a way as to make the trivial solution of the complete closed non-linear system asymptotically stable.

However, in the most common cases, as indicated in Sections 2 and 3, system (1.5), (1.7) is not completely controllable and observable. It is generally impossible, therefore, to guarantee asymptotic stability of a steady motion (1.3) by introducing feedback based on estimation of the state vector. In these case, having constructed a control for the controllable subsystem, one must analyse the stability of the trivial solution of the complete closed non-linear system (Lyapunov's theorem of stability in the first approximation is not applicable in this case).

Suppose conditions 1 and 2 are satisfied. As shown in Section 2, system (1.5) splits into two subsystems. One of them corresponds to the zero roots of the characteristic equation. If the assumptions of Theorems 4–6 hold, subsystem (2.3) is controllable and one can construct a control for it in the form

$$u^{(j)} = -K_{j1}x - K_{j2}\dot{x} - K_{j3}\eta_1, \quad j = 1,2,3 \tag{4.1}$$

where K_{ji} are constant matrices of appropriate orders, chosen subject to the conditions for asymptotic stability of system (2.3) closed by control (4.1).

If the observability conditions of Theorem 8 are satisfied, one can design a linear feedback based on an estimation of the state vector of the system, in the form

$$u^{(j)} = -K_{j1}\tilde{x} - K_{j2}\dot{\tilde{x}} - K_{j3}\tilde{y}, \quad j = 1,2,3 \tag{4.2}$$

$\tilde{x}, \dot{\tilde{x}}, \tilde{y}$ being estimates of the vectors x, $\dot{x}$ and y obtained, for example, from the estimation algorithm

$$\dot{\tilde{w}} = A_w\tilde{w} + L_w(\sigma - C_w\tilde{w}) + B_w u, \quad w = \operatorname{col}\lfloor x^T, \dot{x}^T, y^T, z^T \rfloor \tag{4.3}$$

where $\sigma = C_w w$ is a measurement with respect to which the system is observable. The matrix of the gains L_w is determined based on some criterion for smallness of the estimation error $\Delta w = w = \tilde{w}$. The estimation error Δw must satisfy the equation

$$\Delta\dot{w} = (A_w - L_w C_w)\Delta w$$

whose characteristic polynomial may be prescribed in advance, if the system is observable, by a suitable choice of the constant matrix of the filter gains L_w. In particular, if there are random measurement errors, the matrix L_w can be chosen so as to minimize the variance of the estimation error Δw. The closed controllable system is then described by relations (2.3), (4.2) and (4.3).

Let us investigate the stability of the trivial solution of the complete system (2.1), (2.2), closed by a linear control, on the assumption that subsystem (2.3) is completely controllable and the controls $u^{(1)}$, $u^{(2)}$ and $u^{(3)}$ have the form of (4.2).

The characteristic equation of the linear system (2.1), (2.2) closed by control (4.2) has $m - (k + h_1)$ zero roots, while the others lie in the left half-plane. It can be shown that this system may be reduced to a form which corresponds completely to the special case of several zero roots, and the Lyapunov–Malkin theorem [10, 17], according to which the trivial solution of the system is stable, will hold. Under those conditions any perturbed motion sufficiently close to the unperturbed motion will approach one of the possible SMs as $t \to \infty$.

Note that questions of the stability of the full linear system closed by a linear control affecting only the cyclic coordinates or some of them, when Conditions 1 and 2 are satisfied, have been considered previously [8].

5. EXAMPLE

Let us consider the classical problem of the motion of a Chaplygin sleigh moving along an inclined surface [9]. A heavy rigid body rests on an inclined plane P supported on three knife-edges, two of which are absolutely smooth and the third is equipped with a semicircular blade. The projection of the mass centre of the body onto the plane P lies on a straight line perpendicular to the blade and passing through the point K at which the blade touches the plane P. The generalized coordinates will be the Cartesian coordinates ξ_1 and ξ_2 of the point K (where the ξ_1 axis is parallel to the horizontal plane and the ξ_2 axis is directed upward with respect to the supporting plane P) and the angle of rotation φ of the body about a straight line perpendicular to the plane P. A non-holonomic constraint, expressing the condition that the body does not slide in the direction orthogonal to the plane of the blade, is described by the equation $\dot{\xi}_2 = \dot{\xi}_1 \operatorname{tg}\varphi$. The Lagrangian has the form

$$L = \frac{m}{2}\left[(\dot{\xi}_1 + l\dot{\varphi}\cos\varphi)^2 + (\dot{\xi}_2 + l\dot{\varphi}\sin\varphi)^2 + k^2\dot{\varphi}^2\right] - mg\sin\alpha(\xi_2 - l\cos\varphi)$$

where m is the mass, k is the radius of inertia, α is the inclination of the plane, and l is the distance from the projection of the mass centre onto the plane P to the point K. As has been observed [10], the above constraint is not of Chaplygin type. It can be verified that the equations of motion admit of a steady motion

$$\varphi(t)=\varphi_0\ (\varphi_0=0,\pi),\ \dot{\varphi}(t)=0,\ \dot{\xi}_1=\nu,\ \xi_2=\xi_{20} \tag{5.1}$$

representing uniform translation of the body with velocity ν, with the blade moving parallel to the ξ_1 axis. In this case Condition 1 holds only in steady motion. The equations in Voronets form, linearized in the neighbourhood of this steady motion, have the form (1.5)

$$\rho^2\ddot{x}+l\delta\dot{y}+gl\delta\sin\alpha x=b_1u_1$$
$$l\delta\ddot{x}+\dot{y}+g\sin\alpha x=b_2u_3$$
$$\dot{z}=\nu x$$

where

$$x=\varphi-\varphi_0,\ y=\dot{\xi}_1-\nu,\ z=\xi_2-\xi_{20};\ \delta=\cos\varphi_0,\ \rho^2=l^2+k^2$$

and b_1u_1 and b_2u_3 are the linear parts of the control inputs acting along the positional and cyclic coordinates, respectively.

By Theorem 1, the system is controllable if $b_1b_2\nu\neq0$. This means that the equilibrium position is not controllable ($\nu\neq0$) and that both positional and cyclic coordinates must be subject to controls.

One can then construct a feedback control with respect to all the variables x, $\dot{x}$, y and z which guarantees asymptotic stability of solutions (5.1) for the complete non-linear system of equations of the perturbed motion (by Lyapunov's theorem of stability in the first approximation [17]).

If the control is introduced only along the cyclic coordinate ($u_1\equiv0$), the system is not completely controllable and, introducing variables

$$\zeta=l\delta g\sin\alpha z+\nu(\rho^2\dot{x}+\delta ly),\ \ \eta=l\delta\dot{x}+y$$

one can reduce the system to the form (2.1), (2.2):

$$k^2\ddot{x}=-\delta lu_2,\ \ \dot{\eta}=-g\sin\alpha x+u_2,\ \ \dot{\zeta}=0$$

By Theorem 2, subsystem (2.1) is controllable with respect to the variables $x,\dot{x}$, and η if $l\sin\alpha\neq0$. If a control (4.1) is constructed, the Lyapunov–Malkin theorem [17] will guarantee stability of solutions (5.1) with respect to all the variables x, $\dot{x}$, y and z for the complete linear system of equations of perturbed motion.

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